Quantified Modal Logics and Tableau Methods

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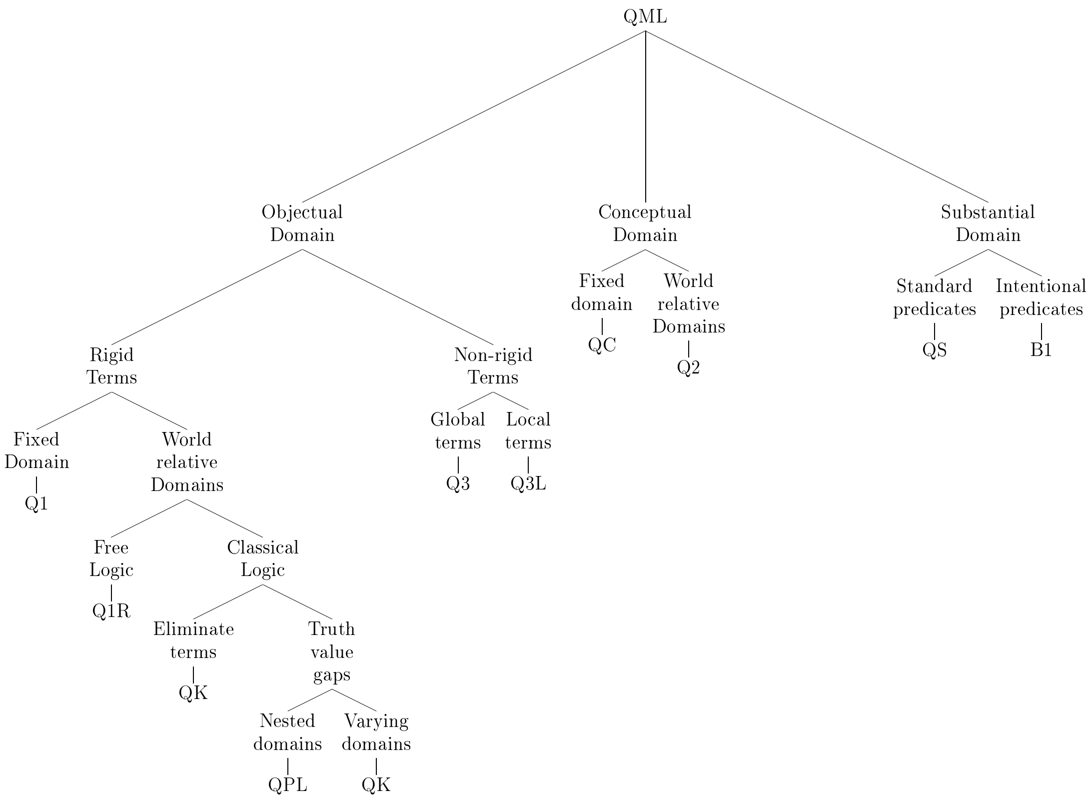
Quantified Modal Logics (QMLs)

"Just add the principles of first-order logic to propositional modal logic"?

Garson (1984):



Possible-world Semantics for QMLs

- Accessibility relation (propositional basis)
- Constraints on the object domains
  - varying domains
  - cumulative (increasing/monotonic) domains
  - constant domains
- Constraints on the designation of terms
  - non-rigid designation
  - rigid designation

Quantifiers range on the object domain associated with each world:

$\forall x A$ is true at w iff A holds for every object in the domain associated with w

$\exists x A$ is true at w iff A holds for some object in the domain associated with w

Modal Interpretations

$\mathcal{M}$ are made of:

- a set W of "possible worlds"
- an "initial world" w_0
- an accessibility relation $R \subseteq W \times W$
- a domain assignment function δ, such that $\delta(w) \neq \emptyset$; the global object domain is some $D \supseteq \bigcup_{w \in W} \delta(w)$
- an interpretation ϕ of functional symbols:
  $\phi(w, f) \in D^k \to D$.
  Requirements on ϕ:
  - if $d_1, ..., d_n \in \delta(w)$, then $\phi(w, f)(d_1, ..., d_n) \in \delta(w)$ (**local terms**)
  - if f is a **rigid** symbol, then for all $w, w' \in W$: $\phi(w, f) = \phi(w', f)$
- an interpretation π of predicate symbols: $\pi(w, p) \subseteq D^k$

Constant domains: for all $w \in W$, $\delta(w) = D$.

Cumulative domains: for all $w, w' \in W$, if wRw' then $\delta(w) \subseteq \delta(w')$.

Varying domains: no restrictions on $\delta(w)$.

The Satisfiability Relation

Variable assignments: $s : X \to D$

s_x^d: assignment like s except that $s_x^d(x) = d$

$s_w(t)$: interpretation of the term t in w according to s (as usual)

1. $\mathcal{M}, w \models_s p(t_1, ..., t_n)$ iff $\langle s_w(t_1), ..., s_w(t_n)\rangle \in \pi(w, p)$
2. $\mathcal{M}, w \models_s \neg A$ iff $\mathcal{M}, w \not\models_s A$
3. $\mathcal{M}, w \models_s A \wedge B$ iff $\mathcal{M}, w \models_s A$ and $\mathcal{M}, w \models_s B$
4. $\mathcal{M}, w \models_s A \vee B$ iff $\mathcal{M}, w \models_s A$ or $\mathcal{M}, w \models_s B$
5. $\mathcal{M}, w \models_s \forall x A$ iff for all $d \in \delta(w)$, $\mathcal{M}, w \models_{s_x^d} A$
6. $\mathcal{M}, w \models_s \exists x A$ iff there exists $d \in \delta(w)$ such that $\mathcal{M}, w \models_{s_x^d} A$
7. $\mathcal{M}, w \models_s \Box A$ iff for all $w' \in W$ such that wRw', $\mathcal{M}, w' \models_s A$
8. $\mathcal{M}, w \models_s \Diamond A$ iff there is a $w' \in W$ such that wRw' and $\mathcal{M}, w' \models_s A$

A is true in $\mathcal{M}$ iff for all variable assignments s on $\mathcal{M}$: $\mathcal{M}, w_0 \models_s A$

Proof Systems for QMLs

Propositional modal system +
classical quantifier axioms/rules

= cumulative domain logics
with rigid designation

In fact, $\Box\forall x A \to \forall x \Box A$ is derivable

Axiomatic approach
(a review in Garson, 1984)

Propositional modal axioms/rules +
classical quantifier axioms/rules +
the Barcan formula ($\forall x \Box A \to \Box\forall x A$)

= constant domain logics
with rigid designation

In other variants, the classical principles governing quantifiers fail

Rigid and non-rigid designation

If terms designate non-rigidly, the rule of instantion fails

$$(INST) \quad \forall x A \to A[t/x]$$

For instance, if $INST$ holds,

$$S = \{p(m(c)), \neg \Box p(m(c)), \forall x(p(x) \to \Box p(x))\}$$

is unsatisfiable, because

$$p(m(c)) \to \Box p(m(c))$$

follows by instantiation from $\forall x(p(x) \to \Box p(x))$

However, if m denotes non rigidly, S is satisfiable: let $\Box$ be the belief operator and

$p(x)$:	x is not Italian
$m(x)$:	the mayor of the city x
c:	New York

(Konolige, 1986)

Proof methods for QMLs

External:

Semantics-based Translation Methods

(Ohlbach, 1991; Auffray & Enjalbert, 1992)

All domain variants, both rigid and non-rigid terms

Internal:

“pure” methods

“impure” methods

(use of prefixes/labels/indeces/ ...)

Pure Methods (I)

(beyond Hilbert-style systems)

Resolution: Abadi & Manna (1986): constant domains, both rigid and non-rigid symbols

(Note that resolution is never easy)

Rewrite rules:

- Simplification rules:
  $q \vee \Diamond(\underline{\neg p} \wedge q \wedge \underline{p}) \Rightarrow q \vee \Diamond(q \wedge false)$
- Deduction rules:
  $q \vee (\underline{\Diamond q} \wedge r \wedge \underline{\Box p}) \Rightarrow q \vee (\Diamond q \wedge r \wedge \Box p \wedge \Diamond(p \wedge q))$

  Flexible Skolemization:

  $$\Box \exists x (p(x) \wedge \Diamond p(x)) \Rightarrow \Box \forall x\, (x = a \rightarrow (p(x) \wedge \Diamond p(x)))$$

  where a is a new non-rigid constant

Pure Methods (II)

Display calculi: (pure?)
multiple structure connectives + "display"

$$\text{LK:} \quad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \qquad \text{DL:} \quad \frac{X \vdash A \circ B}{X \vdash A \vee B}$$

Quantifier rules + modal display calculi = constant domain QMLs (Belnap 1982)
Structure-dependent rules: varying, constant, increasing, decreasing domains (Wansing 1999)

Pure Tableau Methods for QML: "standard" ground tableaux (Fitting 1983)

Classical propositional rules	
$(\alpha)\ \dfrac{A \wedge B, S}{A, B, S}$	$(\beta)\ \dfrac{A \vee B, S}{A, S \qquad\qquad B, S}$

Modal propositional rules (S4)	
$(\nu)\ \dfrac{\Box A, S}{A, \Box A, S}$	$(\pi)\ \dfrac{\Diamond A, \Box S, S'}{A, S, \Box S}$

Classical quantifier rules	
$(\gamma_G)\ \dfrac{\forall x A, S}{A[t/x], \forall x A, S}$ t: a ground term occurring in the branch	$(\delta_G)\ \dfrac{\exists x A, S}{A[c/x], S}$ c: a Skolem constant new to $\{\exists x A\} \cup S$

Dropping Monotonicity: Varying Domains (Fitting 83)

Let's assume there are no functional symbols

With each tableau branch $\mathcal{B}$ is associated the set of constants "alive" on that branch, $\epsilon(\mathcal{B})$:

- $\mathcal{B}$ is the initial tableau: $\epsilon(\mathcal{B})$ is the empty set
- When the δ-rule is applied with Skolem constant c, c is added to $\epsilon(\mathcal{B})$
- When the π-rule is applied, $\epsilon(\mathcal{B})$ becomes the empty set

The γ-rule becomes:

$$\frac{\forall x A, S}{A[c/x], \forall x A, S}$$

where c is a constant alive in the branch

Empty domains are also allowed

The Barcan formula cannot be proved

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\forall x \Box A \to \Box \forall x A)}{\forall x \Box A,\ \neg \Box \forall x A}\ (\alpha)}{\Box A[a/x],\ \forall x \Box A,\ \neg \Box \forall x A}\ (\gamma)}{\Box A[a/x],\ A[a/x],\ \neg \forall x A}\ (\pi)}{\Box A[a/x],\ A[a/x],\ \neg A[c/x]}\ (\delta_G)$$

In the application of the δ_G-rule, a *new* constant must be used

And the universal formula is lost: we have forgotten the past

Constant-domains and symmetry of the accessibility relation

If a model is symmetric and its domains are cumulative, then it is a constant-domain model

Symmetric logics cannot be captured by "standard" tableau methods (cut-free sequent calculi)

The Craig Interpolation Theorem does not hold for constant-domain logics (Fine, 79), so:

> ... there can be no "reasonable" cut-free tableau system for such logics for if there were we could use it, or the related symmetric Gentzen system, to prove an Interpolation result (Fitting 83)

However, constant-domain logics (with a propositional analytical basis) can be given a simple proof-theoretical characterization by means of free-variable tableau systems (Cerrito, Cialdea 2001)

Free-variable Tableaux

(Fitting 1988: increasing domains)

Dynamic Skolemization:

$(\gamma)\ \dfrac{\forall x A, S}{A[v/x], \forall x A, S}$	$(\delta)\ \dfrac{\exists x A, S}{A[f(v_1, ..., v_k)/x], S}$
v: a new free variable	f: a new Skolem function $v_1, ..., v_k$: all the free variables in $\{\exists x A\} \cup S$

MGU Atomic Closure Rule

If:

- $\mathcal{T}$ is a tableau for S
- some branch of $\mathcal{T}$ contains A and $\neg B$, where A and B are atomic
- σ is a MGU of A and B

then the $\mathcal{T}\sigma$ is a tableau for S.

Relaxing the δ-rule

The "liberalized" δ-rule" (δ^+-rule)

> where only the free variables actually occurring in $\exists x A$ are required to be the arguments of the Skolem function

in the classical case, is a more efficient - but equivalent - reformulation of the δ-rule (Hähnle, Shmitt 94)

The δ^+-rule is not sound with respect to cumulative-domain QML.

It makes the Barcan formula provable:

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\forall x \Box A \to \Box \forall x A)}{\forall x \Box A,\ \neg \Box \forall x A}\ (\alpha)}{\Box A[v/x],\ \forall x \Box A,\ \neg \Box \forall x A}\ (\gamma)}{A[v/x],\ \Box A[v/x],\ \neg \forall x A}\ (\pi)}{A[v/x],\ \Box A[v/x],\ \neg A[c/x]}\ (\delta^+)$$

The tableau is closed by application of the substitution $\{c/v\}$

Free-variable Tableau for Constant Domain QMLs

(Cerrito & Cialdea 2001)

$$\frac{\begin{array}{c}\text{Propositional modal rules for a cut-free logic}\\ (\mathbf{K}, \mathbf{D}, \mathbf{T}, \mathbf{K4} \text{ or } \mathbf{S4}) +\\ \text{free-variable } \gamma\text{-rule} + \text{substitution rule}\\ \delta^{+}\text{-rule} =\end{array}}{\begin{array}{c}\text{constant domain QML}\\ \text{(with rigid designation)}\end{array}}$$

Cumulative-domains: unification of $f(v_1, ..., v_k)$, denoting in a domain D_2, with a variable v ranging over the domain D_1 of a "previous" world must be forbidden, because it may be $D_1 \subset D_2$

Constant domains: run-time Skolemization is allowed, but formulae cannot be initially skolemized, because quantifiers cannot always cross the modal operators:

$\forall x \Diamond A \to \Diamond \forall x A$ is not valid

“Impure methods”: structured ground prefixes

Prefixed Tableaux (Fitting 83):

Dynamic rules cause one to forget the past

This is inappropriate for logics in which you can “go home” again

- Symmetric logics
- Constant domain logics

Tableaux with prefixed formulae: the prefix is a “name” for a possible world (Fitting 1972, 1983)

Prefixed formula: σA, where σ is a prefix

σA is true if A holds in the world σ names.

Prefixes: finite sequences of positive integers.
Ex: 1,1,2,1

The structure of prefixes reflects the accessibility relation of the logic: a prefix σ' is “accessible” from σ if σ is a (proper or improper) initial segment of σ' (**S4**)

These methods do not lend themselves to interpolation theorem proofs

Prefixed Tableau Rules (constant domains)

A node in a prefixed tableau is labelled by a prefixed formula.

A tableau proof of A begins with $1\,\neg A$

$$(\alpha)\ \frac{\sigma\, A \wedge B}{\begin{array}{c}\sigma\, A\\ \sigma\, B\end{array}} \qquad\qquad (\beta)\ \frac{\sigma\, A \vee B}{\sigma\, A \qquad \sigma\, B}$$

$$(\nu)\ \frac{\sigma\, \Box A}{\sigma'\, A} \qquad\qquad (\pi)\ \frac{\sigma\, \Diamond A}{\sigma, n\, A}$$

if σ' is accessible from σ

where σ, n is "new" to the branch

Quantifier rules for constant domain logics:

$$(\gamma)\ \frac{\sigma\, \forall x A}{\sigma\, A[t/x]} \qquad\qquad (\delta)\ \frac{\sigma\, \exists x A}{\sigma\, A[c/x]}$$

where t is any ground term

where c is a constant new to the branch

A proof of the Barcan formula

$$
\cfrac{1\,\neg(\forall x \Box A \to \Box \forall x A)}{\begin{array}{c} 1\,\neg\Box\forall x A \\ 1\,\forall x \Box A \\ \hline 1,1\,\neg\forall x A \\ \hline 1,1\,\neg A[c/x] \\ \hline 1\,\Box A[c/x] \\ \hline 1,1\,A[c/x] \end{array}
\begin{array}{l} \\ \\ (\pi) \\ (\delta) \\ (\gamma) \\ (\nu) \end{array}}\;(\alpha)
$$

Since the past is not forgotten, the discovery that something (c) exists in a world $(1,1)$ makes one realize that it must have existed in the world we left behind (1)

Other Domain Variants

Each prefix has an associated set of Skolem constants, distinct prefixes having disjoint associated sets

Consider the simple case where the only ground terms are Skolem constants

The δ-rule:

$$\frac{\sigma\, \exists x A}{\sigma\, A[c/x]}\ (\delta)$$

where c is a new Skolem constant associated with σ

The γ-rule:

$$\frac{\sigma\, \forall x A}{\sigma\, A[c/x]}\ (\gamma)$$

where c is associated to:

Cumulative domains	**Varying domains**
any prefix σ' such that σ is accessible from σ'	σ

"Impure methods": prefixes made of variables and Skolem terms + unification

Sequent systems: Jackson & Reichgelt(1987)

Constant domain QMLs

Indexed formulae: indexes represent worlds

$A_{f(w,x):w:0}$: for every world w accessible from 0 and for every object x there is a world accessible from w where A is true

Formulae unification takes indexes into account and is logic-dependent

Resolution: Ohlbach (1988): constant domains. Atoms are indexed by "world-paths": sequences of variables and Skolem terms. Unification acts on world-paths, too.

The matrix method (Wallen, 1990)
(all domain variants)

A generalization of Bibel's connection-method

Formulae are indexed by prefixes: their role is similar in prefixed tableaux (the methods are inter-translatable). Prefixed formulae are analysed in the style of ground tableaux

A prefix contain both constants and variables, to which substitutions can be applied (the "correctness" of a substitution depends on the accessibility relation of the logic)

Impure Methods: atomic world names

Labelled Natural Deduction Systems

(Basin, Matthews, Viganò, 1998) (internal?)

Rigid designation, all domain variants

- Base logic: labelled formulae $w : A$
- Relational theory: accessibility relation wRw'
- Domain theory: $w : t$ (t exists at world w; similar to $E(t)$ in free logic)

$$\frac{\begin{array}{c}[w : t] \\ \vdots \\ w : A[t/x]\end{array}}{w : \forall x\, A} \qquad \frac{w : \forall x\, A \quad w : t}{w : A[t/x]}$$

Domains can be empty, terms may not designate

Increasing domains: $\dfrac{w_i R w_j \quad w_i : t}{w_j : t}$

Decreasing domains: $\dfrac{w_i R w_j \quad w_j : t}{w_i : t}$

Impure methods: annotations on terms

Resolution methods:

Geissler & Konolige (1986), **Konolige** (1986):

cumulative domains, non-rigid terms

The language is extended with a "bullet operator", used to instantiate variables in a modal context:

$$\forall x(p(x) \to \Box p(x)) \Rightarrow p(m(c)) \to \Box p(\bullet m(c))$$

$\bullet t$: the designation of t in the actual world

Skolemization does not affect quantifiers in a modal context:

$$\forall x \exists y(\neg p(x,y) \vee \Box \exists z\, q(x,y,z)) \Rightarrow$$
$$\neg p(x, f(x)) \vee \Box \exists z\, q(\bullet x, \bullet f(x), z)$$

There are recursive calls to the theorem prover in order to determine whether two formulae can be resolved against each other

Cialdea & Fariñas del Cerro (1986), **Cialdea** (1991): cumulative domains, rigid terms

Modal Skolemization: free variables and functional symbols are annotated by an integer representing their "modal depth":

$$\forall x \Box \exists y p(x, y) \Rightarrow \Box p(x^0, f^1(x^0))$$

Modal substitutions: a variable labelled by n can be replaced only by terms containing only symbols s^m with $m \leq n$

Impure methods: atomic world names + annotations on terms

A unified approach to QMLs:
(Cerrito & Cialdea 2000, 2001)

A language can contain both rigid and non-rigid functional symbols

A rigid functional symbol f designates the same function in every possible world. A non-rigid f may designate different functions in different worlds.

Free-variable tableaux:

- tableau nodes are labelled by $n : S$ where n is a positive integer (the "name" of a possible world) and S a set of annotated formulae.
- functional symbols can be annotated by a natural number $(f^0, f^1, ...)$: the "name" of the world where they are meant to designate

If f is a rigid symbol, it occurs as f^0 in the tableau

If f is non-rigid, it may occur as $f^1, f^2, ...$

Initial Tableau

$$1 : S^*$$

where S^* is obtained from S by annotating:

- rigid symbols with 0
- non-modal occurrences of non-rigid symbols with 1

Example: if f is rigid and c, g are non-rigid:

$$1 : \forall x(p(f^0(x, c^1), g^1(x)) \wedge \Box \exists y\, q(g(f^0(c, y))))$$

Propositional Rules

$$(\alpha)\ \frac{n : A \wedge B, S}{n : A, B, S} \qquad (\beta)\ \frac{n : A \vee B, S}{n : A, S \qquad\qquad n : B, S}$$

$$(\nu)\ \frac{n : \Box A, S}{n : A^n, \Box A, S} \qquad (\pi)\ \frac{n : \Diamond A, \Box S, S'}{m : A^m, S^m, \Box S}$$

where $m \in \mathbb{N}$ is new in the whole tableau and $m > n$

Quantifier Rules

Depth: relation between annotated terms and positive integers. "t is at depth n": its intended designation belongs to the domain of the world named by n.

t is at depth n if t contains only symbols annotated with:

varying domains: n and 0
cumulative domains: $k \leq n$
constant domains: any k

A free variable v^n **ranges** over depth k iff:

varying and cumulative domains: $k = n$
constant domains: $k \in \mathbb{N}$

$$(\gamma)\ \frac{n : \forall x A, S}{n : A[v^n/x], \forall x A, S} \qquad v\text{: a new free variable}$$

$$(\delta^+)\ \frac{n : \exists x A, S}{n : A[f^n(v_1^{k_1}, ..., v_m^{k_m})/x], S}$$

f: a Skolem function new to the tableau
$v_1^{k_1}, ..., v_m^{k_m}$: all the free variables occurring in $\exists x A$ and ranging over n

Substitution Rule

As usual, but:

Modal substitutions are:

$\sigma = \{t_1/v_1^{k_1}, ..., t_m/v_m^{k_m}\}$, where t_i is at depth k_i

Example 1: the Barcan Formula

$$
\cfrac{\cfrac{\cfrac{\cfrac{1 : \neg(\forall x \Box A \to \Box \forall x A)}{1 : \forall x \Box A,\ \neg \Box \forall x A}\ (\alpha)}{1 : \Box A[v^1/x],\ \forall x \Box A,\ \neg \Box \forall x A}\ (\gamma)}{2 : A[v^1/x],\ \neg \forall x A, \Box A[v^1/x]}\ (\pi_4)}{2 : A[v^1/x],\ \neg A[c^2/x],\ \Box A[v^1/x]}\ (\delta^+)
$$

Constant domains: the tableau is closed by application of the substitution $\{c^2/v^1\}$.

Varying or cumulative domains: $\{c^2/v^1\}$ is not a modal substitution (c^2 is not at depth 1)

Example 2

(the agent's beliefs about the mayor of New York)

c: rigid $\quad$ m: non-rigid

$S = \{p(m(c)), \forall x(p(x) \to \Box p(x)), \neg\Box p(m(c))\}$ is satisfiable

$$
\cfrac{\cfrac{\begin{array}{l} 1 : p(m^1(c^0)),\ \neg\Box p(m(c^0)) \\ \quad \forall x(p(x) \to \Box p(x)), \end{array}}{\begin{array}{l} 1 : p(m^1(c^0)),\ \neg\Box p(m(c^0)) \\ \quad p(v^1) \to \Box p(v^1), \\ \quad \forall x(p(x) \to \Box p(x)) \end{array}}\ (\gamma)}{\begin{array}{l} 1 : p(m^1(c^0)),\ \neg p(v^1), \\ \quad \forall x(p(x) \to \Box p(x)), \\ \quad \neg\Box p(m(c^0)) \end{array} \qquad \cfrac{\begin{array}{l} 1 : p(m^1(c^0)),\ \Box p(v^1), \\ \quad \forall x(p(x) \to \Box p(x)), \\ \quad \neg\Box p(m(c^0)) \end{array}}{\begin{array}{l} 2 : p(v^1),\ \neg p(m^2(c^0)), \\ \quad \Box p(v^1) \end{array}}\ (\pi)}\ (\beta)
$$

The tableau cannot be closed: the unification problem $\{p(m^1(c^0)) = p(v^1),\ p(v^1) = p(m^2(c^0))\}$ has no solution.

Other Tableau Systems for QMLs

Fitting 1999 (prefixed tableaux: varying domains, non-rigid terms).

Non-rigid designation is captured by means of predicate abstraction:

$$\langle \lambda x. \Diamond p(x) \rangle (c) \qquad \Diamond \langle \lambda x. p(x) \rangle (c)$$

This allows modal Skolemization:

$$\begin{array}{lcl} \Box \exists x p(x) & \Rightarrow & \Box \langle \lambda x. p(x) \rangle (c) \\ \exists x \Box p(x) & \Rightarrow & \langle \lambda x. \Box p(x) \rangle (c) \end{array}$$

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