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Acyclically 3-Colorable Planar Graphs

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ABSTRACT

In this paper we study the planar graphs that admit an acyclic 3-coloring. We show that testing acyclic 3-colorability is $\mathcal{NP}$-hard, even for planar graphs of maximum degree 4, and we show that there exist infinite classes of cubic planar graphs that are not acyclically 3-colorable. Further, we show that every planar graph has a subdivision with one vertex per edge that admits an acyclic 3-coloring. Finally, we show that every series-parallel graph admits an acyclic 3-coloring and we give a linear-time algorithm for recognizing whether every 3-coloring of a series-parallel graph is acyclic.

1 Introduction

A *coloring* of a graph is an assignment of *colors* to the vertices such that no two adjacent vertices have the same color. A *k-coloring* is a coloring using k colors. Planar graph colorings have been widely studied from both a combinatorial and an algorithmic point of view. The existence of a 4-coloring for every planar graph, proved by Appel and Haken [3, 4], is one of the most famous results in Graph Theory. A quadratic-time algorithm is known to compute a 4-coloring of any planar graph [14].

An *acyclic coloring* is a coloring with *no bichromatic cycle*. An *acyclic k-coloring* is an acyclic coloring using k colors. Acyclic colorings have been deeply investigated in the literature. From an algorithmic point of view, Kostochka proved in [11] that deciding whether a graph admits an acyclic 3-coloring is $\mathcal{NP}$-hard. From a combinatorial point of view, the most interesting result is perhaps the one proved by Alon *et al.* in [2], namely that every graph with degree Δ can be acyclically colored with $O(\Delta^{4/3})$ colors, while there exist graphs requiring $\Omega(\Delta^{4/3}/\sqrt[3]{\log \Delta})$ colors in any acyclic coloring.

Acyclic colorings of planar graphs have been first considered in 1973 by Grünbaum, who proved in [9] that there exist planar graphs requiring 5 colors in any acyclic coloring. Moreover, the same lower bound holds even for 3-degenerate bipartite planar graphs [12]. Grünbaum conjectured that such a bound is tight and proved that 9 colors suffice for constructing such a coloring. The Grünbaum upper bound was improved down to 8 [13], to 7 [1], to 6 [10], and finally to 5 by Borodin [5].

Since there exist planar graphs requiring 5 colors in any acyclic coloring, it is natural to study which planar graphs can be acyclically 3- or 4-colored. In this paper we study the acyclically 3-colorable planar graphs, from both an algorithmic and a combinatorial perspective. We show the following results:

- In Sect. 3 we prove that deciding whether a planar graph has an acyclic 3-coloring is an $\mathcal{NP}$-hard problem, even when restricted to planar graphs of degree 4. An $\mathcal{NP}$-hardness proof for deciding acyclic 3-colorability was only known for non-planar graphs of unbounded degree [11], as far as we know. The $\mathcal{NP}$-hardness result is not surprising, since an analogous result is known for deciding (possibly non-acyclic) 3-colorability of planar graphs of degree 4 [8]. However, we show an interesting difference between the class of 3-colorable planar graphs and the class of acyclically 3-colorable planar graphs, by exhibiting an infinite number of cubic planar graphs not admitting any acyclic 3-coloring (while K_4 is the only cubic graph that can not be 3-colored [7]). We remark that it is known how to construct acyclic 4-colorings of every cubic (even non-planar) graph [15].

- In Sect. 4 we prove that every planar graph has a subdivision with one vertex per edge that is acyclically 3-colorable. Such a result complements the observation that every planar graph has a subdivision with one vertex per edge that is 2-colorable. Acyclic colorings of graph subdivisions have been already considered by Wood in [17], where the author observed that every graph has a subdivision with two vertices per edge that is acyclically 3-colorable.

- In Sect. 5 we prove that every series-parallel graph has an acyclic 3-coloring, thus improving the result of Grünbaum [9] that every outerplanar graph has an acyclic 3-coloring. Further, we consider the problem of determining the planar graphs such that every 3-coloring is acyclic. Such a problem has been introduced by Grünbaum [9], who showed that every 3-coloring of a maximal outerplanar graph is acyclic. We improve his result by characterizing the series-parallel graphs such that every 3-coloring is acyclic and by providing a linear-time recognition algorithm.

2 Preliminaries

A graph G is *k-connected* if removing any k-1 vertices leaves G connected; 3-connected and 2-connected graphs are called *triconnected* and *biconnected* graphs, respectively. The *degree of a vertex* is the number of incident edges. The *degree of a graph* is the maximum degree of one of its vertices. In a *cubic* graph (resp. a *subcubic* graph) each vertex has degree exactly 3 (resp. at most 3). A *subdivision* of a graph G is obtained by replacing each edge of G with a path. A *k-subdivision* of G is a subdivision of G in which any path replacing an edge of G has at most k internal vertices. The internal (extremal) vertices of the paths replacing the edges of G are called *subdivision vertices* (resp. *main vertices*).

A *planar graph* is a graph containing no K_5-minor and no $K_{3,3}$-minor. A planar graph is *maximal* when all its faces are delimited by 3-cycles.

An *outerplanar graph* is a graph that admits a planar drawing in which all the vertices are incident to the outer face. Combinatorially, an outerplanar graph is a graph containing no K_4-minor and no $K_{2,3}$-minor. An outerplanar graph is *maximal* if all its internal faces are delimited by 3-cycles.

A *series-parallel graph (SP-graph for short)* is a graph containing no K_4-minor. SP-graphs are inductively defined as follows. An edge (u, v) is a SP-graph with *poles* u and v. Denote by u_i and v_i the poles of a SP-graph graph G_i. A *series composition* of a sequence $G_0, G_1, \dots, G_k$ of SP-graphs, with $k \geq 1$, is a SP-graph with poles u=u_0 and v=v_k, containing graphs G_i as subgraphs, and such that v_i and u_{i+1} have been identified, for each i=$0, 1, \dots, k-1$. A *parallel composition* of a set $G_0, G_1, \dots, G_k$ of SP-graphs, with $k \geq 1$, is a SP-graph with poles u=u_0=u_1=$\dots$=u_k and v=v_0=v_1=$\dots$=v_k and containing graphs G_i as subgraphs. The SPQ-tree $\mathcal{T}$ of a SP-graph G is the tree, rooted at any node, representing the series and parallel compositions of G.

3 Deciding the Acyclic 3-Colorability of Planar Graphs

In this section we study the problem of deciding whether a given planar graph admits an acyclic 3-coloring. We have the following:

Theorem 1 *Planar Graph Acyclic 3-Colorability is $\mathcal{NP}$-complete.*

Proof: The problem is clearly in $\mathcal{NP}$. In order to show the $\mathcal{NP}$-hardness, we perform a reduction from Planar Graph 3-Colorability. Consider the graph G_9 shown in Fig. 1.a. We claim that any acyclic 3-coloring of G_9 satisfies the following properties: (P1) u_1 and u_2 have different colors; (P2) every path connecting u_1 and u_2 contains vertices of all the three colors.

We prove the claim. Assume $c(u_1)$=c_0. Since v_1 and v_2 are adjacent to u_1, either $c(v_1)$=$c(v_2)$=c_1, or $c(v_1)$=c_1 and $c(v_2)$=c_2. Suppose that $c(v_1)$=$c(v_2)$=c_1. Then, $c(v_3)$=c_2, since $c(v_3)\neq c_0$ (otherwise cycle $(u_1, v_1, v_3, v_2, u_1)$ would be bichromatic) and $c(v_3)\neq c_1$ (v_3 is adjacent to v_1). Further, $c(v_4)$=c_0 (v_4 is adjacent to v_2 and v_3) and $c(v_5)$=c_1 (v_5 is adjacent to v_3 and v_4). Then, there is no possible coloring for v_6. In fact, $c(v_6)\neq c_0$ (otherwise cycle $(u_1, v_2, v_4, v_5, v_6, v_1, u_1)$ would

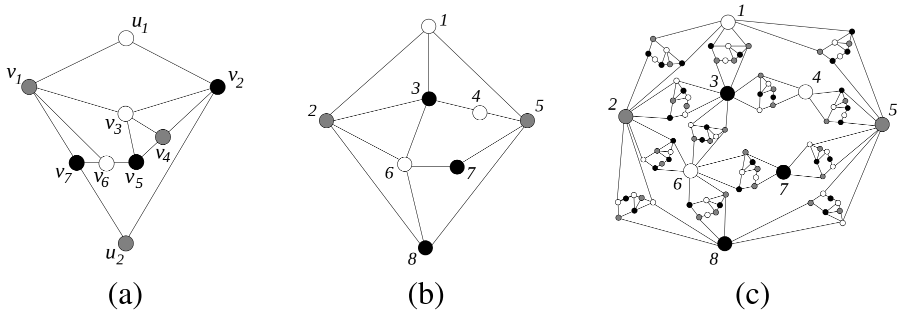


Figure 1: (a) Graph G_9 and its unique acyclic 3-coloring, up to a switch of the color classes. (b) A planar graph G. (c) The planar graph G' obtained by replacing each edge of G with a copy of G_9.

be bichromatic), $c(v_6){\neq}c_1$ (v_6 is adjacent to v_5), $c(v_6){\neq}c_2$ (otherwise cycle $(v_1, v_3, v_5, v_6, v_1)$ would be bichromatic). Hence, in any acyclic 3-coloring of G_9, $c(v_1)$=c_1 and $c(v_2)$=c_2. Then, $c(v_3)$=c_0 (v_3 is adjacent to v_1 and v_2), $c(v_4)$=c_1 (v_4 is adjacent to v_2 and v_3), $c(v_5)$=c_2 (v_5 is adjacent to v_3 and v_4), $c(v_6)$=c_0 (v_6 is adjacent to v_1 and v_5), and $c(v_7)$=c_2 (v_7 is adjacent to v_1 and v_6). Finally, $c(u_2)$=c_1, since $c(u_2){\neq}c_0$ (otherwise cycle $(u_2, v_7, v_6, v_5, v_3, v_2, u_2)$ would be bichromatic) and $c(u_2){\neq}c_2$ (u_2 is adjacent to v_2). Hence, G_9 has only one acyclic 3-coloring (up to a switch of the color classes), which satisfies properties P1 and P2.

We reduce Planar Graph 3-Colorability to Planar Graph Acyclic 3-Colorability. Let G be an instance of Planar Graph 3-Colorability (see Fig. 1.b). Replace each edge (u, v) of G with a copy of G_9, by identifying vertices u and v with u_1 and u_2, respectively (see Fig. 1.c). Let G' be the resulting planar graph. We show that G admits a 3-coloring if and only if G' admits an acyclic 3-coloring.

First, suppose that G admits a 3-coloring. For each edge (u, v) of G, let c_0 and c_1 be the colors of u and v, respectively. Color the corresponding graph G_9 by assigning color c_0 to u_1, color c_1 to u_2, and by completing the unique acyclic 3-coloring of G_9 with c_0 and c_1. We show that the resulting coloring of G' is acyclic. Assume, for a contradiction, that G' contains a bichromatic cycle $\mathcal{C}$. Such a cycle is not entirely contained inside a graph G_9 replacing an edge of G in G' (in fact, the 3-coloring of each graph G_9 is acyclic). Hence, $\mathcal{C}$ contains vertices of more than one graph G_9. This implies that $\mathcal{C}$ contains as a subgraph a simple path connecting vertices u_1 and u_2 of a graph G_9. However, by property P2 of the G_9's coloring, such a path contains vertices of all the three colors, a contradiction.

Second, suppose that G' admits an acyclic 3-coloring. A coloring of G is obtained from the acyclic 3-coloring of G' by assigning to each vertex u of G the color of the corresponding vertex u_1 of G'. By property P1, each edge of G connects two vertices of distinct colors. □

Next, we show that the problem of testing whether a planar graph admits an acyclic 3-coloring remains $\mathcal{NP}$-hard even when restricted to planar graphs of maximum degree 4.

Theorem 2 *Degree-4 Planar Graph Acyclic 3-Colorability is $\mathcal{NP}$-complete.*

Proof: The problem is clearly in $\mathcal{NP}$. In order to show the $\mathcal{NP}$-hardness, we perform a reduction from Planar Graph Acyclic 3-Colorability. Consider the family of graphs H_i defined as follows. H_1 is shown in Fig. 2.a. H_i is obtained from a copy of H_{i-1} and a copy of H_1 by renaming vertices u_1, v_1, and w_1 of H_1 with labels u_i, v_i, and w_i, respectively, and by identifying vertex w_{i-1} of H_{i-1} and vertex u_i of H_1. H_3 is shown in Fig. 2.b. Vertices u_j, v_j, and w_j of H_i, for $1 \leq j \leq i$, are the *outlets* of H_i. The family of graphs H_i has been defined in [8] in order to perform a reduction from *Planar Graph Colorability* to *Degree-4 Planar Graph Colorability*. We claim that H_i satisfies the following properties: (P0) H_i admits an acyclic 3-coloring; (P1) in any acyclic 3-coloring of H_i, the outlets have the same color; (P2) in any acyclic 3-coloring of H_i, for any two outlets x_j and y_k of H_i, there exist two bichromatic paths connecting x_j and y_k, one with colors c_0 and c_1, and one with colors c_0 and c_2, respectively, where $x, y \in \{u, v, w\}$, $j, k \in \{1, 2, \ldots, i\}$, and c_0 is the color of the outlets.

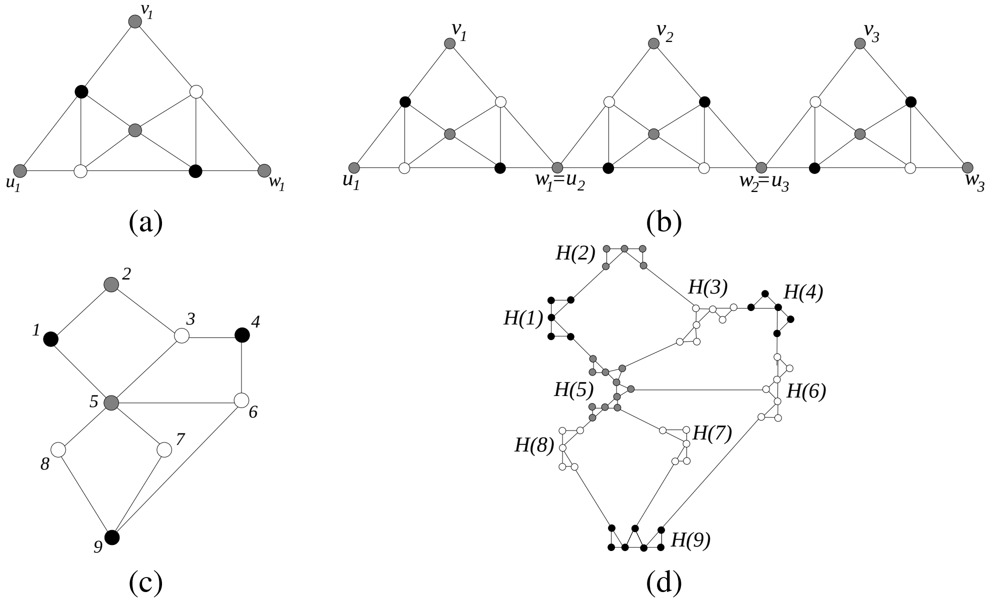


Figure 2: (a) Graph H_1. (b) Graph H_3. (c) A planar graph G. (d) Graph G' obtained by replacing each degree-d vertex z of G with a copy $H(z)$ of H_d. For each graph $H(z)$, only its outlets are shown.

We prove the claim. A property stronger than P1 was proved in [8], where in fact it is shown that in any 3-coloring of H_i the outlets have the same color. We prove P0 and P2 by induction on i. P0 and P2 are easily verified in H_1, namely Fig. 2.a shows the unique acyclic 3-coloring of H_1, up to a switch of the color classes. Suppose that P0 is verified in H_{i-1}. Every cycle of H_i entirely belongs either to H_{i-1} or to the copy of H_1 that is added to H_{i-1} to form H_i. In both cases the cycle is not bichromatic, by induction. Suppose that P2 is verified in H_{i-1}. Consider any two outlets x_j and y_k of H_i. If $x_j, y_k \notin \{v_i, w_i\}$ (if $x_j, y_k \in \{v_i, w_i\}$), by induction x_j and y_k are connected by two bichromatic paths with colors c_0 and c_1, and with colors c_0 and c_2, respectively. If $x_j \notin \{v_i, w_i\}$ and $y_k \in \{v_i, w_i\}$, x_j and y_k are connected by a bichromatic path with colors c_0 and c_1 (resp. c_0 and c_2), obtained as the union of a bichromatic path with colors c_0 and c_1 (resp. c_0 and c_2) between x_j and u_i and a bichromatic path with colors c_0 and c_1 (resp. c_0 and c_2) between u_i and y_k. All such paths exist by induction.

We reduce Planar Graph Acyclic 3-Colorability to Degree-4 Planar Graph Acyclic 3-Colorability. Let G be any instance of Planar Graph Acyclic 3-Colorability (Fig. 2.c). For each vertex z of G with d neighbors $z_1, z_2, \ldots, z_d$, delete z and its incident edges from G, introduce a copy of $H(z)$=H_d, and add an edge between outlet v_j of $H(z)$ and z_j, for each j=$1, 2, \ldots, d$ (Fig. 2.d). We show that the resulting planar graph G' of degree 4 admits an acyclic 3-coloring if and only if G' admits an acyclic 3-coloring.

Suppose that G admits an acyclic 3-coloring. Color the outlets z_j corresponding to each vertex z of G with the color of z. By properties P0 and P1, the coloring of each $H(z)$ can be completed to an acyclic 3-coloring. Any cycle $\mathcal{C}'$ of G' either is entirely contained in a graph $H(z)$ (hence $\mathcal{C}'$ is not bichromatic), or contains vertices of several graphs $H(z)$. In the latter case, partition the vertices of $\mathcal{C}'$ into sets $V_1, V_2, \dots, V_k$, where each V_j is a maximal sequence of consecutive vertices of $\mathcal{C}'$ belonging to the same graph $H(z)$. Suppose, for a contradiction, that $\mathcal{C}'$ is bichromatic. Consider the (possibly non-simple) cycle $\mathcal{C}$ of G containing a vertex z if $\mathcal{C}'$ passes through vertices of $H(z)$, and containing an edge (z_1, z_2) if $\mathcal{C}'$ contains an edge between a vertex of $H(z_1)$ and a vertex of $H(z_2)$. If $\mathcal{C}$ contains vertices of three colors, then $\mathcal{C}'$ contains vertices of three colors since, for each vertex z of G, the outlets of $H(z)$ have the same color of z. However, $\mathcal{C}'$ is supposed to be bichromatic, hence $\mathcal{C}$ is bichromatic, as well, contradicting the assumption that the coloring of G is acyclic.

Suppose that G' admits an acyclic 3-coloring. Color G by assigning to each vertex z the color of the outlets of $H(z)$ (by P1, all such outlets have the same color). Suppose that G contains a bichromatic cycle $\mathcal{C}$ with colors c_0 and c_1. A bichromatic cycle $\mathcal{C}'$ in G' is found as follows: Replace each vertex z_1 of $\mathcal{C}$ with a path with colors c_0 and c_1 connecting the outlets of $H(z_1)$ adjacent to the outlets of $H(z_2)$ and $H(z_3)$, where z_2 and z_3 are the neighbors of z_1 in $\mathcal{C}$. Such a path exists by Property P2. Then, $\mathcal{C}'$ is a bichromatic cycle in G', contradicting the assumption that the coloring of G' is acyclic. □

Now we show infinite classes of cubic planar graphs not admitting any acyclic 3-coloring. Such a result is based on the following lemmata. The proof of Lemma 2 is analogous to the proof of Lemma 1. Denote by $K_{2,3}$ the complete bipartite graph whose vertex sets $V^A_{2,3}$ and $V^B_{2,3}$ have two and three vertices, respectively. Denote by $K_{1,1,2}$ the complete tripartite graph whose vertex sets $V^A_{1,1,2}$, $V^B_{1,1,2}$, and $V^C_{1,1,2}$ have one, one, and two vertices, respectively.

Lemma 1 *Let G be a graph having a vertex z of degree 2 adjacent to two vertices u and v. Let G' be the graph obtained by substituting z with a copy of $K_{2,3}$, where a vertex $u^B_{2,3}$ of $V^B_{2,3}$ is connected to u and a vertex $v^B_{2,3} \neq u^B_{2,3}$ of $V^B_{2,3}$ is connected to v (see Fig. 3.a and Fig. 3.b). Then G' has an acyclic 3-coloring if and only if G has an acyclic 3-coloring.*

Proof: Suppose that G has an acyclic 3-coloring. Color each vertex of G' not in $K_{2,3}$ as in G, the vertices in $V^B_{2,3}$ with $c(z)$, and the vertices in $V^A_{2,3}$ with the two colors different from $c(z)$. Every cycle $\mathcal{C}'$ in G' not passing through the vertices of $K_{2,3}$ is also a cycle in G (hence it is not bichromatic). Every cycle $\mathcal{C}'$ in G' passing through vertices of $K_{2,3}$ contains a path $\mathcal{P}'$ from u to v whose vertices belong to $K_{2,3}$. Suppose, for a contradiction, that $\mathcal{C}'$ is bichromatic. Path $\mathcal{P}'$ contains a vertex in $V^B_{2,3}$ with color $c(z)$. The cycle $\mathcal{C}$ of G obtained by replacing $\mathcal{P}'$ with path (u, z, v) in $\mathcal{C}'$ is bichromatic, a contradiction.

Now suppose that G' has an acyclic 3-coloring. In any acyclic coloring of $K_{2,3}$, the three vertices in $V^B_{2,3}$ have the same color c_0. Color each vertex of G different from z as in G' and color z with c_0. Every cycle $\mathcal{C}$ in G that does not pass through z is also a cycle in G' (hence it is not bichromatic). Every cycle $\mathcal{C}$ in G that passes through z contains path (u, v, z). Suppose, for a contradiction, that all the vertices of $\mathcal{C}$ have colors c_0 and c_1. For each color c_i, with $i \in \{1, 2\}$, there exists a path $\mathcal{P}_i$ connecting u and v and whose vertices belong to $K_{2,3}$ and have colors c_0 and c_i. The cycle $\mathcal{C}'$ of G' obtained by replacing (u, z, v) with path $\mathcal{P}_1$ in $\mathcal{C}$ is bichromatic, a contradiction. □

Lemma 2 *Let G be a graph having a vertex z of degree 2 adjacent to two vertices u and v. Let G' be the graph obtained by substituting z with a copy of $K_{1,1,2}$, where a vertex $u^C_{1,1,2}$ of $V^C_{1,1,2}$*

is connected to u and a vertex $v^C_{1,1,2}\neq u^C_{1,1,2}$ of $V^C_{1,1,2}$ is connected to v (see Fig. 3.a and Fig. 3.c). Then G' has an acyclic 3-coloring if and only if G has an acyclic 3-coloring.

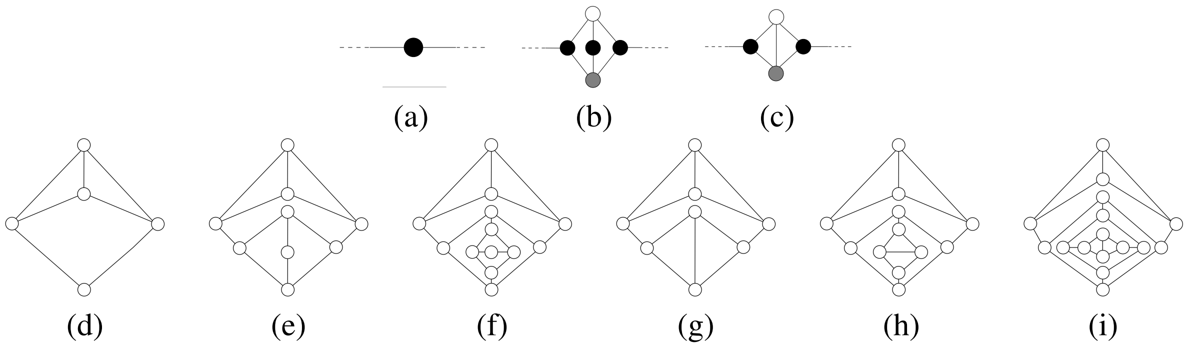


Figure 3: (a) and (b) Replacement of a degree-2 vertex with a $K_{2,3}$. (a) and (c) Replacement of a degree-2 vertex with a $K_{1,1,2}$. (d) G_5. (e) G_9. (f) G_{13}. (g) G^+_5. (h) G^+_9. (i) G^+_{13}.

Graph G_5 (Fig. 3.d) has no acyclic 3-coloring and has a vertex of degree 2. For every $i > 0$, replace the vertex of degree 2 of graph G_{4i+1} with a copy of $K_{2,3}$, obtaining a graph G_{4i+5} that has a vertex of degree 2 and, by Lemma 1, is not acyclically 3-colorable. Figs. 3.e–f show G_9 and G_{13}. Replacing the vertex of degree 2 of G_{4i+1} with a copy of $K_{1,1,2}$ yields a graph G^+_{4i+1} that, by Lemma 2, is not acyclically 3-colorable. Figs. 3.g–i show graphs G^+_5, G^+_9, G^+_{13}. Graphs G^+_{4i+1} are cubic, for every $i > 0$.

4 Acyclic 3-Colorings of Planar Graph Subdivisions

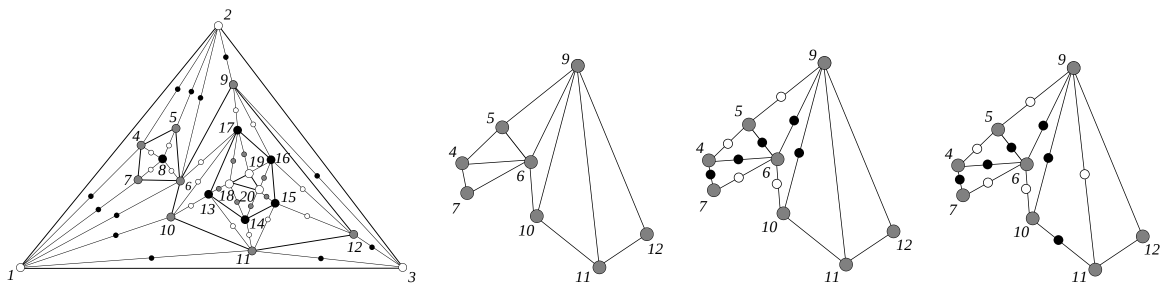
In this section we prove the following theorem.

Theorem 3 *Every planar graph has a 1-subdivision that admits an acyclic 3-coloring.*

Proof: It suffices to prove the statement for maximal planar graphs. In fact, suppose that the statement holds for maximal planar graphs. Let G be a planar graph. Augment G to a maximal planar graph G' by adding dummy edges. Then G' has a 1-subdivision G'_s that has an acyclic 3-coloring c. Remove the edges of G'_s corresponding to subdivided dummy edges of G', obtaining a planar graph G_s that is a subdivision of G. Since every cycle of G_s is also a cycle of G'_s, c is an acyclic 3-coloring of G_s.

Consider a planar drawing of any maximal planar graph G. Let G_s be the planar graph obtained by subdividing each edge of G with one subdivision vertex. Partition the vertices of G into disjoint sets $V^0, V^1, \dots, V^k$ as follows. Let G^0=G; till there are vertices in G^i, denote by V^i the main vertices incident to the outer face of G^i; remove the vertices in V^i and their incident edges from G^i obtaining a graph G^{i+1}. Notice that the vertices in each set V^i induce an outerplanar subgraph of G. Further, each edge of G is either incident to two vertices in the same set V^i or to two vertices in sets V^i and V^{i+1}, for some $i \in \{0, 1, \dots, k-1\}$.

Color the main vertices in V^i with color $c_{j(i)}$, where $j(i) \in \{0, 1, 2\}$ and $j(i) \equiv i \bmod 3$. Color each subdivision vertex adjacent to a vertex in a set V^i and to a vertex in V^{i+1} with color $c_{j(i+2)}$. See Fig. 4.a. It remains to color each subdivision vertex adjacent to two vertices belonging to the same set V^i. Consider the outerplanar subgraph O^i of G induced by the vertices in V^i. Augment O^i to maximal by adding dummy edges. See Fig. 4.b. Let O^i_s be the plane graph obtained by subdividing each edge of O^i with one subdivision vertex. Each subdivision vertex of G_s adjacent to two vertices belonging to the same set V^i, for some $i \in \{1, 2, \ldots, k\}$, is also a subdivision vertex of O^i_s. Hence, a coloring of the subdivision vertices of O^i_s determines a coloring of each subdivision vertex of G_s adjacent to two vertices in the same set V^i. We



(a) (b) (c) (d)

Figure 4: (a) Coloring the main vertices and the subdivision vertices of G_s adjacent to a vertex in a set V^i and to a vertex in V^{i+1}. Thick edges connect vertices of G in the same set V^i. (b) Subgraph O^2 of G augmented to maximal. (c)–(d) Coloring O^2_s at step x and at step $x+1$ of the algorithm. Not yet colored subdivision vertices of O^2_s are not shown.

show how to color the subdivision vertices of O^i_s. The algorithm already chose to color the main vertices of O^i_s with color $c_{j(i)}$. Since O^i is maximal, every internal face of O^i_s has three subdivision vertices. The coloring algorithm exploits several steps. At the first step, consider any internal face f^* of O^i_s. Color two of its subdivision vertices with color $c_{j(i+1)}$ and the third one with $c_{j(i+2)}$. At the x-th step, with $x \geq 2$, suppose that the subgraph $O^{i,x}_s$ of O^i_s induced by the colored subdivision vertices and by their neighbors is biconnected. See Fig. 4.c. Consider any internal face of O^i_s of which one subdivision vertex has already been colored. Color the other two subdivision vertices incident to the face, one with color $c_{j(i+1)}$ and the other one with $c_{j(i+2)}$. See Fig. 4.d.

We show that the resulting coloring of G_s is acyclic. Suppose, for a contradiction, that a bichromatic cycle $\mathcal{C}$ exists. If $\mathcal{C}$ contains main vertices in two distinct sets V^i and V^{i+1}, then $\mathcal{C}$ contains two edges (v_p, v_s) and (v_s, v_q), where v_p and v_q are main vertices in V^i and V^{i+1}, respectively, and v_s is a subdivision vertex. However, $c(v_p)$=$c_{j(i)}$, $c(v_q)$=$c_{j(i+1)}$, and $c(v_s)$=$c_{j(i+2)}$, hence $\mathcal{C}$ is not bichromatic, a contradiction. Otherwise, $\mathcal{C}$ only contains main vertices in the same set V^i. Then, $\mathcal{C}$ is also a cycle of O^i_s. We show by induction that the described coloring of O^i_s is acyclic. The coloring of f^* is acyclic. Suppose that, after a certain step of the coloring algorithm for the vertices of O^i_s, the subgraph $O^{i,x}_s$ of O^i_s induced by the colored subdivision vertices and by their neighbors is acyclic. When a new face is considered and two subdivision vertices v_1 and v_2 are colored, with $c(v_1)$=$c_{j(i+1)}$ and $c(v_2)$=$c_{j(i+2)}$, every cycle either entirely belongs to $O^{i,x}_s$, hence by induction it is not bichromatic, or passes through v_1, v_2, and their common neighbor, hence it is not bichromatic, a contradiction. □

5 Acyclic 3-Colorings of Series-Parallel Graphs

In this section we consider acyclic 3-colorings of SP-graphs. First, we show that every SP-graph admits an acyclic 3-coloring.

Theorem 4 *Every SP-graph G with poles u and v admits an acyclic 3-coloring such that $c(u) \neq c(v)$ and every path connecting u and v, except for edge (u, v), contains a vertex w with $c(w) \neq c(u), c(v)$.*

Proof: We prove the statement by induction on the number n of vertices. Case n=2 is trivial. Suppose $n > 2$ and distinguish two cases: (Case 1) G is a series composition of SP-graphs $G_0, G_1 \cdots, G_k$, such that G_i has poles u_i and v_i, with u_0=u, v_i=u_{i+1}, and v_k=v; (Case 2) G is a parallel composition of SP-graphs $G_0, G_1 \cdots, G_k$ with poles u and v.

In Case 1, apply induction to construct an acyclic 3-coloring of G_i with colors c_0, c_1, and c_2 such that $c(u_i)$=$c_{j(i)}$ and $c(v_i)$=$c_{j(i+1)}$, for each i=$0, 1, \ldots, k-1$, where $j(i) \in \{0, 1, 2\}$ and $j(i) \equiv i \bmod 3$. Further, apply induction to construct an acyclic 3-coloring of G_k with colors c_0, c_1, and c_2 such that $c(u_k)$=$c_{j(k)}$, and such that $c(v_k)$=c_1, if $c(u_k)$=c_0 or $c(u_k)$=c_2, and $c(v_k)$=c_2, if $c(u_k)$=c_1. By construction, $c(u_0$=$u)$=c_0, $c(u_1)$=c_1, $c(u_2)$=c_2, and every path connecting u and v passes through u_0, u_1, and u_2, hence it is not bichromatic. Further, any simple cycle in G is also a cycle in a component G_i hence, by induction, the coloring of G is acyclic.

In Case 2, apply induction to construct an acyclic 3-coloring of G_i, for i=$0, 1, \cdots, k$, with colors c_0, c_1, and c_2 such that $c(u)$=c_0, $c(v)$=c_1, and every path connecting u and v in G_i, except for edge (u, v), contains a vertex w with $c(w)$=c_2. By construction, $c(u)$=c_0 and $c(v)$=c_1. Further, every path connecting u and v is also a path in a component G_i which, by induction, contains a vertex with color c_2, unless it is edge (u, v). Let $\mathcal{C}$ be any simple cycle in G. If all the vertices of $\mathcal{C}$ belong to a graph G_i, then $\mathcal{C}$ is not bichromatic by induction. Otherwise, $\mathcal{C}$ contains vertices u and v, hence it consists of two paths $\mathcal{P}_1$ and $\mathcal{P}_2$ connecting u and v and belonging to two distinct components G_i and G_j. At most one of $\mathcal{P}_1$ and $\mathcal{P}_2$, say $\mathcal{P}_1$, coincides with edge (u, v). By induction, $\mathcal{P}_2$ contains a vertex of color c_2. □

Now we turn our attention to determine which are the SP-graphs such that every 3-coloring is acyclic. A characterization and a linear-time algorithm are obtained through the following lemmata, which characterize the SP-graphs that satisfy some coloring properties described below.

First, we characterize the SP-graphs that have a 3-coloring in which the poles have distinct colors and in which the poles have the same color.

Corollary 1 *Every SP-graph with poles u and v admits a 3-coloring with $c(u) \neq c(v)$.*

Lemma 3 *Every SP-graph G with poles u and v admits a 3-coloring with $c(u)$=$c(v)$ if and only if G does not contain edge (u, v).*

Proof: The necessity is trivial. We inductively prove the sufficiency. Suppose that G is a parallel composition of SP-graphs $G_0, G_1, \ldots, G_k$ and that G does not contain edge (u, v). Then, no component G_i contains (u, v), hence it admits a 3-coloring in which $c(u)$=$c(v)$ by induction. Suppose that G is a series composition of graphs $G_0, G_1, \ldots, G_k$. Color G_0 so that $c(u)$=c_0 and the other pole of G_0 has color c_1. Such a coloring exists by Corollary 1. For $1 \leq i \leq k-1$, assume that the color of the pole that G_i shares with G_{i-1} has been already determined to be either c_1 or c_2. Color the pole that G_i shares with G_{i+1} with color c_2 or c_1, respectively, and color G_i so that its poles have colors c_1 and c_2 (such a coloring exists by Corollary 1). Complete the coloring of G by setting $c(v)$=c_0 and by coloring G_k so that its poles have colors c_0 and either c_1 or c_2. Again, such a coloring exists by Corollary 1. □

Second, we characterize the SP-graphs that have a 3-coloring in which there exists a bichromatic path between the poles. The proof of Lemma 5 is analogous to the proof of Lemma 4.

Lemma 4 *Every SP-graph G with poles u and v admits a 3-coloring with $c(u){\neq}c(v)$ and with a bichromatic path between u and v if and only if the following two conditions are satisfied:*

1. *if G is a parallel composition of SP-graphs, then there exists a component that admits a 3-coloring with $c(u){\neq}c(v)$ and with a bichromatic path between u and v;*
2. *if G is a series composition of SP-graphs $G_0, G_1, \ldots, G_k$, then each component admits a 3-coloring with a bichromatic path between its poles, and at least one of the following holds:*
   - *(a) there exists a component G_i with poles u_i and v_i that admits a 3-coloring with $c(u_i){=}c(v_i)$ and with a bichromatic path between u_i and v_i, and a 3-coloring with $c(u_i){\neq}c(v_i)$ and with a bichromatic path between u_i and v_i;*
   - *(b) there exists an odd number of components that admit a 3-coloring in which the poles have different colors and are connected by a bichromatic path.*

Proof: We prove the necessity. If G is a parallel composition of SP-graphs and Condition 1 does not hold, then every 3-coloring in which $c(u){\neq}c(v)$ does not contain a bichromatic path between u and v. Suppose that G is a series composition of SP-graphs $G_0, G_1, \ldots, G_k$ and that Condition 2 does not hold, that is, there exists a component G_i with poles u_i and v_i that admits no 3-coloring with a bichromatic path between u_i and v_i. Every path connecting u and v contains a path connecting u_i and v_i, hence it is not bichromatic. Suppose that G is a series composition of graphs $G_0, G_1, \ldots, G_k$, that Condition 2 holds, and that neither Condition 2a nor Condition 2b holds. Then, in every 3-coloring with a bichromatic path between u and v, there is an even number of components G_i such that $c(u_i){\neq}c(v_i)$ and hence $c(u){=}c(v)$.

We prove the sufficiency. Case $G = (u, v)$ is trivial. If G is a parallel composition of SP-graphs, by Condition 1 there exists a component that admits a 3-coloring with $c(u){\neq}c(v)$ and with a bichromatic path between u and v. By Corollary 1, all other components can be colored so that $c(u){\neq}c(v)$. If G is a series composition of SP-graphs $G_0, G_1, \ldots, G_k$ and Conditions 2 and 2a hold, set $c(u_0){=}c_0$. For $0 \leq j \leq i-1$, assume that $c(u_j)$ has already been determined to be either c_0 or c_1; color G_j so that there exists a bichromatic path between u_j and v_j and so that $c(v_j)$ is either c_0 or c_1. Analogously, set $c(v_k){=}c_1$. For $k \geq j \geq i+1$, assume that $c(v_j)$ has been determined to be either c_0 or c_1; color G_j so that there exists a bichromatic path between u_j and v_j and so that $c(u_j)$ is either c_0 or c_1. Color G_i so that there exists a bichromatic path between u_i and v_i; this can be done both if $c(u_i){=}c(v_i)$ and if $c(u_i){\neq}c(v_i)$. Finally, if G is a series composition of SP-graphs and Conditions 2 and 2b hold, then each component has either a 3-coloring with a bichromatic path between its poles and the poles have the same color, or a 3-coloring with a bichromatic path between its poles and the poles have distinct colors. Color each component with such a coloring, so that its poles have colors in $\{c_0, c_1\}$. Since an odd number of components have poles with different colors, $c(u){\neq}c(v)$. □

Lemma 5 *Every SP-graph G with poles u and v admits a 3-coloring with $c(u){=}c(v)$ and with a bichromatic path between u and v if and only if the following three conditions are satisfied:*

1. *G does not contain edge (u, v);*
2. *if G is a parallel composition of SP-graphs, then there exists a component admitting a 3-coloring with $c(u){=}c(v)$ and with a bichromatic path between u and v;*

3. *if G is a series composition of SP-graphs $G_0, G_1, \ldots, G_k$, then each component admits a 3-coloring with a bichromatic path between its poles, and at least one of the following holds:*

   (a) *there exists a component G_i with poles u_i and v_i admitting a 3-coloring with $c(u_i)$=$c(v_i)$ and with a bichromatic path between u_i and v_i, and a 3-coloring with $c(u_i)\neq c(v_i)$ and with a bichromatic path between u_i and v_i;*

   (b) *there exists an even number of components admitting a 3-coloring in which the poles have different colors and are connected by a bichromatic path.*

Third, we characterize the SP-graphs such that every 3-coloring in which the poles have distinct colors is acyclic.

Lemma 6 *Let G be a SP-graph with poles u and v. Suppose that G is a parallel composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then every 3-coloring of G with $c(u)\neq c(v)$ is acyclic if and only if the following two conditions are satisfied:*

1. *for each component G_i, every 3-coloring with $c(u)\neq c(v)$ is acyclic;*
2. *there exist no two components admitting a 3-coloring with $c(u)\neq c(v)$ and with a bichromatic path between u and v.*

Proof: We prove the necessity. If Condition 1 does not hold, a component G_i exists that admits a non-acyclic 3-coloring with $c(u)\neq c(v)$. If Condition 2 does not hold, two components exist that admit a 3-coloring with $c(u)\neq c(v)$ and with a bichromatic path between u and v. Such paths form a bichromatic cycle in G. In both cases, by Corollary 1, each not yet colored component of G admits a 3-coloring with $c(u)\neq c(v)$. Hence, a non-acyclic 3-coloring of G can be constructed.

We prove the sufficiency. Consider any 3-coloring of G such that $c(u)\neq c(v)$. Every cycle in G is either entirely contained inside a component of G (and then it is not bichromatic, by Condition 1), or it consists of two paths between the poles of G. However, one of such paths is not bichromatic (by Condition 2). □

Lemma 7 *Let G be a SP-graph with poles u and v. Suppose that G is a series composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then every 3-coloring of G with $c(u)\neq c(v)$ is acyclic if and only if every 3-coloring of each component G_i is acyclic.*

Proof: We prove the necessity. Suppose that a component G_i that admits a non-acyclic 3-coloring exists with $c(u_i)\neq c(v_i)$ (resp. with $c(u_i)$=$c(v_i)$). If $i < k$, construct any 3-coloring of G_j, where $j\neq i$ and $j < k$. Let c_0 and c_x be the colors of u and u_k, where $x \in \{0, 1\}$. Construct a 3-coloring of G_k with $c(u_k)$=c_x and $c(v_k)$=c_2. Such a coloring exists by Corollary 1. The resulting 3-coloring of G has $c(u)\neq c(v)$ and is not acyclic. If i=k, a non-acyclic 3-coloring of G with $c(u)\neq c(v)$ can be constructed analogously by first coloring G_j, with $j > 0$, and by then suitably coloring G_0.

We prove the sufficiency. Consider any 3-coloring of G with $c(u)\neq c(v)$. Each cycle in G is entirely contained inside a component of G and then it is not bichromatic. □

Fourth, we characterize the SP-graphs such that every 3-coloring in which the poles have the same color is acyclic. The proofs of Lemmata 8 and 9 are analogous to the proofs of Lemmata 6 and 7.

Lemma 8 *Let G be a SP-graph with poles u and v. Suppose that G is a parallel composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then every 3-coloring of G with $c(u)$=$c(v)$ is acyclic if and only if one of the following two conditions is satisfied:*

1. *there exists a component G_i not admitting any 3-coloring with $c(u_i)$=$c(v_i)$;*
2. *for each component G_i, every 3-coloring with $c(u)$=$c(v)$ is acyclic and no two components exist admitting a 3-coloring with $c(u)$=$c(v)$ and with a bichromatic path between u and v.*

Lemma 9 *Let G be a SP-graph with poles u and v. Suppose that G is a series composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then every 3-coloring of G with $c(u)$=$c(v)$ is acyclic if and only if the following three conditions are satisfied:*

1. *for each component G_i with poles u_i and v_i, every 3-coloring with $c(u_i) \neq c(v_i)$ is acyclic;*
2. *if $k > 2$, for each component G_i with poles u_i and v_i, every 3-coloring with $c(u_i)$=$c(v_i)$ is acyclic;*
3. *if k=2, for each component G_i with poles u_i and v_i, every 3-coloring with $c(u_i)$=$c(v_i)$ is acyclic, or there exists a component not admitting any 3-coloring in which $c(u_i)$=$c(v_i)$.*

Finally, we conclude by observing that a SP-graph with poles u and v is such that every 3-coloring is acyclic if and only if every 3-coloring in which $c(u) \neq c(v)$ is acyclic and every 3-coloring in which $c(u)$=$c(v)$ is acyclic. The above characterization gives rise to a linear-time recognition algorithm:

Theorem 5 *There exists a linear-time algorithm for deciding whether a SP-graph is such that every 3-coloring is acyclic.*

Proof: The SPQ-tree $\mathcal{T}$ of a SP-graph G can be computed in linear-time (see, e.g., [16]). Then, each node μ of $\mathcal{T}$ with poles u_μ and v_μ can be equipped with values indicating whether: (i) $G(\mu)$ admits a 3-coloring with $c(u_\mu)$=$c(v_\mu)$; (ii) $G(\mu)$ admits a 3-coloring with $c(u_\mu) \neq c(v_\mu)$ and with a bichromatic path between u_μ and v_μ, $G(\mu)$ admits a 3-coloring with $c(u_\mu)$=$c(v_\mu)$ and with a bichromatic path between u_μ and v_μ, and $G(\mu)$ admits a 3-coloring with a bichromatic path between u_μ and v_μ; and (iii) every 3-coloring of $G(\mu)$ in which $c(u_\mu) \neq c(v_\mu)$ is acyclic, every 3-coloring of $G(\mu)$ in which $c(u_\mu)$=$c(v_\mu)$ is acyclic, and every 3-coloring of $G(\mu)$ is acyclic. Due to Lemmata 3–9, the computation of such values for μ only requires simple checks on analogous values for the children of μ in $\mathcal{T}$. □

6 Conclusions

In this paper we have shown several results on the acyclic 3-colorability of planar graphs.

We have shown that recognizing acyclic 3-colorable planar graphs is $\mathcal{NP}$-hard, even when restricted to planar graphs of degree 4. Further, we have shown infinite classes of subcubic and cubic planar graphs with no acyclic 3-coloring, result contrasting with the fact that all planar graphs with degree 3 have a 3-coloring, except for K_4 [7]. However, the following problem is still open.

Which is the time complexity of testing whether a sub-cubic graph (resp. a cubic graph) admits an acyclic 3-coloring?

The problem seems to be interesting even when restricted to *triconnected* cubic planar graphs. Moreover, we are aware of only three graphs that are cubic, triconnected, and not acyclic 3-colorable (see Fig. 5). The graphs depicted in Figs. 5.a and 5.b were already known to have no acyclic 3-coloring. On the other hand, the graph depicted in Fig. 5.c seems to have gone unnoticed in the literature.

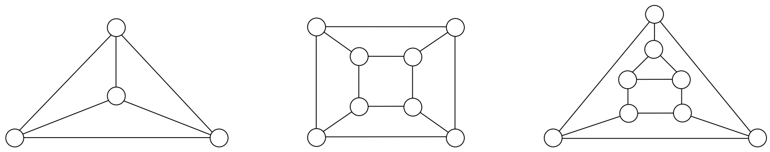


Figure 5: Triconnected cubic planar graphs with no acyclic 3-coloring.

Does an infinite number of triconnected, cubic, and not acyclic 3-colorable planar graphs exist? Which is the time complexity of testing whether a triconnected cubic planar graph admits an acyclic 3-coloring?

We have shown that, for any SP-graph G, whether every 3-coloring is acyclic can be tested in linear time. Testing the same property for general planar graphs (and characterizing the planar graphs for which every 3-coloring is acyclic) seems to be interesting and non-trivial.

Is it possible to test in polynomial time whether every 3-coloring of a given planar graph is acyclic?

Finally, we would like to remind a problem that has been already studied in the literature but that has not been tackled in this paper.

Which is the smallest k such that all planar graphs with girth at least k are acyclic 3-colorable?

Currently, the best known lower bound for k is 5 (the second graph of Fig. 5, proposed by Grünbaum, has girth 4 and is not acyclic 3-colorable [9]). On the other hand, the best known upper bound for k is 7, proved by Borodin, Kostochka, and Woodall [6].

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